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NONLINEAR ANALYSIS OF THICK PLATES ON AN ELASTIC FOUNDATION BY HT FE WITH p-EXTENSION CAPABILITIES

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Abstract—The paper is concerned with the development of hybrid-Trefftz (HT) p-element for nonlinear analysis of Reissner-Mindlin plates resting on an elastic foundation. The foundation may be of Winkler-type or Pasternak-type. Exact solutions of the Lame-Navier equations are used for the in-plane intraelement displacement field and an incremental form of the basic equations is adopted. With the aid of incremental form of these equations, all nonlinear terms may be taken as pseudo-loads. Moreover, some modifications have been made on the nonlinear boundary equations to simplify the ensuing derivation. As a result, the in-plane and out-of-plane equations are uncoupled, and then the derivation for the HT finite element (FE) formulation becomes very simple. The practical efficiency of the new element model has been assessed through several examples. Copyright © 1996 Elsevier Science Ltd

NOTATION

C	$5Et/12(1+v)$ for a homogeneous plate, $G_c(h+t)$ for a sandwich plate
G_W	a part of boundary $\partial\Omega$ of the solution domain Ω, on which deflection W is prescribed, C_{M_n}, C_R etc. are defined similarly
D	$Et^3/12(1-v^2)$ for a homogeneous plate, $E(h+t)^2t/2(1-v^2)$ for a sandwich plate
E	modulus of elasticity
G	$E/2(1+v)$
G_c	core shear modulus
G_p	shear modulus of Pasternak-type foundation
k_p	reaction coefficient of Pasternak-type foundation
k_w	reaction coefficient of Winkler-type foundation
h	core thickness
M_{ij}	bending moment
M_{ij}	twisting moment ($i \neq j$)
N_{ij}	membrane force
n_i	components of the outward normal to the boundary $\partial\Omega$
q	lateral distributed load
Q_i	transverse shear force
r	$(x^2+y^2)^{1/2}$
R	$Q_i n_i + N_n W_{,n} + N_{ns} W_{,s}$
s_i	components of the tangent to the boundary $\partial\Omega$
t	plate thickness (or face-sheet thickness)
U_i	in-plane displacements
W	lateral deflection
δ	variational symbol
δ_{ij}	Kronecker delta
θ	arctg (y/x)
λ^2	$10/t^2$ for a homogeneous plate, $4(1+v)G_c/E(h+t)t$ for a sandwich plate
v	Poisson's ratio
∇^2	$\partial^2/\partial x^2 + \partial^2/\partial y^2$
φ_i	the average rotations normal to the plate mid-surface

1. INTRODUCTION

The hybrid-Trefftz finite element model initiated in 1977 (Jirousek and Leon, 1977; Jirousek, 1978) is now well-established. The method has been widely used in plane elasticity (Jirousek and Teodorescu, 1982; Jirousek and Venkatesh, 1992), plate bending (Jirousek and Guex, 1986; Jirousek and N'Diaye, 1990; Qin, 1994), shells (Voros and Jirousek, 1991), axisymmetric solid mechanics (Wroblewski *et al.* 1992), Poisson's equation (Zielinski and Zienkiewicz, 1985) and heat transfer (Jirousek and Qin, 1995). Further, the p-extension of HT elements was formed in 1982 (Jirousek and Teodorescu, 1982) and has already been found to be particularly advantageous from both the computational point of view and facilities for use. Recently, a detailed assessment on the properties of p-elements has been done (Jirousek and Venkatesh, 1989, 1990; Jirousek *et al.*, 1993). As far as we know, however, there are very few results by HT FE approach for nonlinear problems.

This paper aims at developing a HT p-element model for nonlinear analysis of Reissner plates on an elastic foundation which may be of Winkler-type or Pasternak-type. The main difference between these two foundations is whether the effect of shear interactions is included. By way of an incremental form of the basic equations and some modifications on nonlinear boundary equations, the in-plane and out-of-plane equations are uncoupled. As a consequence, the solution procedure for nonlinear plates becomes quite simple. At the end of the paper, several numerical examples are considered to verify the suitability of the method.

2. GOVERNING EQUATIONS AND THEIR TREFFTZ FUNCTIONS

2.1. *Basic equations*

Consider a Reissner-Mindlin plate of uniform thickness t, 2-D region Ω bounded by its boundary $\partial\Omega$ and resting on an elastic foundation. Throughout this paper repeated indices i, j and k take values in the range $\{1, 2\}$. The nonlinear behavior of the plate is governed by the following equations (Qin, 1993)

$$L_{11}U_1 + L_{12}U_2 - P_1 = 0 \tag{1}$$

$$L_{21}U_1 + L_{22}U_2 - P_2 = 0 \tag{2}$$

$$L_{33}W + L_{34}\varphi_1 - L_{35}\varphi_2 - P_3 = 0 \tag{3}$$

$$L_{43}W + L_{44}\varphi_1 - L_{45}\varphi_2 = 0 \tag{4}$$

$$L_{53}W + L_{54}\varphi_1 - L_{55}\varphi_2 = 0 \tag{5}$$

with

$$\begin{aligned}
&L_{11}() = ()_{,xx} + d_1()_{,yy}, \\
&L_{12}() = L_{21}() = d_2()_{,xy}, \\
&L_{22}() = ()_{,yy} + d_1()_{,xx} \\
&L_{34}() = -L_{43}() = -C()_{,x}, \quad L_{35}() = -L_{53}() = -C()_{,y} \\
&L_{44} = DL_{11} - C, \quad L_{45} = L_{54} = DL_{12}, \\
&L_{55} = DL_{22} - C, \quad U_1 = U_x, \quad U_2 = U_y, \quad \varphi_1 = \varphi_x, \\
&\varphi_2 = \varphi_y, \quad d_1 = (1-\nu)/2, \quad d_2 = (1+\nu)/2 \\
&L_{33} = \begin{cases} C\nabla^2 + k_w & \text{for Winkler-type foundation,} \\ (C - G_p)\nabla^2 + k_p & \text{for Pasternak-type foundation} \end{cases}
\end{aligned}$$

in which a comma followed by a subscript indicates partial differentiation with respect to that subscript, and P_1, P_2 and P_3 are components of pseudo-distributed load defined by (Qin and Huang, 1990; Huang *et al.*, 1992).

$$P_1 = -W_{,x}(W_{,xx}+d_1 W_{,yy})-d_2 W_{,y} W_{,xy}$$

$$P_2 = -W_{,y}(W_{,yy}+d_1 W_{,xx})-d_2 W_{,x} W_{,xy}$$

$$P_3 = J\{(U_{1,x}+0.5W_{,x}^2)(W_{,xx}+vW_{,yy})+(U_{2,y}+0.5W_{,y}^2)(W_{,yy}+vW_{,xx})\}$$
$$+J(1-v)(U_{1,y}+U_{2,x}+W_{,x}W_{,y})W_{,xy}+q$$

where

$$J = \frac{Et}{1-v^2}$$

and the boundary conditions are

$$U_n = U_i n_i = \bar{U}_n \,(\text{on } C_{U_n}), \quad U_s = U_i s_i = \bar{U}_s \,(\text{on } C_{U_s}),$$
$$\varphi_n = \varphi_i n_i = \bar{\varphi}_n \,(\text{on } C_{\varphi_n}), \quad \varphi_s = \varphi_i s_i = \bar{\varphi}_s \,(\text{on } C_{\varphi_s}) \tag{6}$$

$$W = \bar{W}(\text{on } C_W),$$
$$N_n = N_n^l + N_n^n = \bar{N}_n \,(\text{on } C_{N_n}), \tag{7}$$

$$N_{ns} = N_{ns}^l + N_{ns}^n = \bar{N}_{ns} \,(\text{on } C_{N_{ns}}),$$
$$M_n = M_{ij}n_in_j - \alpha G_p W = \bar{M}_n \,(\text{on } C_{M_n}), \quad M_{ns} = M_{ij}n_is_j = \bar{M}_{ns} \,(\text{on } C_{M_{ns}}), \tag{8}$$

$$R = R^l + R^n = \bar{R}\,(\text{on } C_R),$$
$$N_n^l = N_{ij}^l n_i n_j, \quad N_n^n = N_{ij}^n n_i n_j, \quad N_{ns}^l = N_{ij}^l n_i s_j, \quad N_{ns}^n = N_{ij}^n n_i s_j,$$
$$R^l = Q_i n_i, \quad R^n = N_n W_{,n} + N_{ns} W_{,s}$$
$$(\partial\Omega = C_{U_n} \cup C_{N_n} = C_{U_s} \cup C_{N_{ns}} = C_W \cup C_R = C_{\varphi_n} \cup C_{M_n} = C_{\varphi_s} \cup C_{M_{ns}}) \tag{9}$$

where overbar means the prescribed value, and $\alpha = 1$ for the Pasternak-type foundation, $\alpha = 0$ for the Winkler one.

Finally, the constitutive relationships are given by

$$N_{ij} = N_{ij}^n + N_{ij}^l$$
$$N_{ij}^l = Gt\left\{U_{i,j} + U_{j,i} + \frac{2v}{1-v}U_{k,k}\delta_{ij}\right\}, \tag{10}$$

$$N_{ij}^n = Gt\left\{W_{,i}W_{,j} + \frac{v}{1-v}W_{,k}W_{,k}\delta_{ij}\right\}, \tag{11}$$

$$M_{ij} = \frac{1-v}{2}D\left\{\varphi_{i,j} + \varphi_{j,i} + \frac{2v}{1-v}\varphi_{k,k}\delta_{ij}\right\}, \tag{12}$$

$$Q_i = C(W_{,i} - \varphi_i) \tag{13}$$

Noting that eqns (1)–(5) are not, in general, suitable for HT FE analysis, an incremental form of the equations must therefore be adopted to linearize these nonlinear equations. Denoting the incremental variable by the superimposed dot and omitting those infinitesimal terms resulting from the product of incremental variables, one obtains

$$L_{11}\dot{U}_1+L_{12}\dot{U}_2 = \dot{P}_1 \tag{14}$$

$$L_{21}\dot{U}_1+L_{22}\dot{U}_2 = \dot{P} \tag{15}$$

$$L_{33}\dot{W}+L_{34}\dot{\varphi}_1+L_{35}\dot{\varphi}_2 = \dot{P}_3 \tag{16}$$

$$L_{43}\dot{W}+L_{44}\dot{\varphi}_1+L_{45}\dot{\varphi}_2 = 0 \tag{17}$$

$$L_{53}\dot{W}+L_{54}\dot{\varphi}_1+L_{55}\dot{\varphi}_2 = 0 \tag{18}$$

together with

$$\dot{U}_1 = \dot{U}_i n_i = \Delta\bar{U}_n \,(\text{on } C_{U_n}), \quad \dot{U}_s = \dot{U}_i s_i = \Delta\bar{U}_s \,(\text{on } C_{U_s}), \tag{19}$$

$$\dot{\varphi}_n = \dot{\varphi}_i n_i = \Delta\bar{\varphi}_n \,(\text{on } C_{\varphi_n}), \quad \dot{\varphi}_s = \dot{\varphi} s_i = \Delta\bar{\varphi}_s \,(\text{on } C_{\varphi_s}),$$

$$\dot{W} = \Delta\bar{W} (\text{on } C_W),$$

$$\dot{N}_n = N^l_{ij} n_i n_j = (\Delta\bar{N}_n - \dot{N}^n_{ij} n_i n_j) = \bar{N}^*_n (\text{on } C_{N_n}), \tag{20}$$

$$\dot{N}_{ns} = \dot{N}^l_{ij} n_i s_j = (\Delta\bar{N}_{ns} - \dot{N}^n_{ij} n_i s_j) = \bar{N}^*_{ns} \,(\text{on } C_{N_{ns}}),$$

$$\dot{R} = \dot{Q}_1 n_i = (\Delta\bar{R} - \dot{R}^n) = \bar{R}^* \,(\text{on } C_R)$$

$$\dot{M}_n = \dot{M}_{ij} n_i n_j - \alpha G_p \dot{W} = \Delta\bar{M}_n \,(\text{on } C_{M_n}) \tag{21}$$

$$M_{ns} = \dot{M}_{ij} n_i s_j = \Delta\bar{M}_{ns} \,(\text{on } C_{M_{ns}}), \tag{22}$$

where (see Qin and Huang, 1990)

$$\begin{aligned}
\dot{P}_1 &= -\dot{W}_{,x}(W_{,xx}+d_1 W_{,yy}) - W_{,x}(\dot{W}_{,xx}+d_1 \dot{W}_{,yy}) - d_2 \dot{W}_{,y} W_{,xy} - d_2 W_{,y} \dot{W}_{,xy} \\
\dot{P}_2 &= -\dot{W}_{,y}(W_{,yy}+d_1 W_{,xx}) - W_{,y}(\dot{W}_{,yy}+d_1 \dot{W}_{,xx}) - d_2 \dot{W}_{,x} W_{,xy} - d_2 W_{,x} \dot{W}_{,xy} \\
\dot{P}_3 &= J\{(\dot{U}_{1,x} + W_{,x}\dot{W}_{,x})(W_{,xx}+vW_{,yy}) + (U_{1,x}+0.5W^2_{,x})(\dot{W}_{,xx}+v\dot{W}_{,yy}) \\
&\quad + (\dot{U}_{2,y} + W_{,y}\dot{W}_{,y})(W_{,yy}+vW_{,xx}) + (U_{2,y}+0.5W^2_{,y})(\dot{W}_{,yy}+v\dot{W}_{,xx})\} \\
&\quad + J(1-v)\{(\dot{U}_{1,y} + \dot{U}_{2,x} + \dot{W}_{,x}W_{,y} + W_{,x}\dot{W}_{,y})W_{,xy} + (U_{1,y} + U_{2,x} + W_{,x}W_{,y})\dot{W}_{,xy}\} + \dot{q}
\end{aligned}$$

It should be pointed out that $\dot{N}^n_{ij} \ll \dot{N}^l_{ij}$, $\dot{R}^n \ll \dot{R}^l$ in practical problems. So we may move these nonlinear terms to the right-hand side of the above boundary equations. In this way, the in-plane and out-of-plane boundary equations are uncoupled. As a result, the ensuing derivation becomes quite simple, but an iterative approach is required to evaluate the nonlinear terms $\dot{P}_i$, $\bar{N}^*_n$, $\bar{N}^*_{ns}$ and $\bar{R}^*$.

2.2. *Trefftz functions*

The Trefftz functions play an important role in the derivation of HT finite element formulation. In this subsection, the construction of Trefftz functions for Reissner plates on elastic foundations will be discussed in detail. The Trefftz functions of eqns (14) and (15) can be generated in a systematic way from Muskhelishvili's complex variable formulation (Jirousek and Venkatesh, 1992 ; Green and Zerna, 1968). For the reader's convenience, we list those results as follows (Jirousek and Venkatesh, 1992) :

$$\mathbf{U}_j^* = \begin{Bmatrix} \operatorname{Re} Z_{1k} \\ \operatorname{Im} Z_{1k} \end{Bmatrix} \quad \text{with } Z_{1k} = (3-\nu)iz^k + (1+\nu)kiz\bar{z}^{k-1} \tag{23}$$

$$\mathbf{U}_{j+1}^* = \begin{Bmatrix} \operatorname{Re} Z_{2k} \\ \operatorname{Im} Z_{2k} \end{Bmatrix} \quad \text{with } Z_{2k} = (3-\nu)z^k - (1+\nu)kz\bar{z}^{k-1} \tag{24}$$

$$\mathbf{U}_{j+2}^* = \begin{Bmatrix} \operatorname{Re} Z_{3k} \\ \operatorname{Im} Z_{3k} \end{Bmatrix} \quad \text{with } Z_{3k} = (1-\nu)i\bar{z}^k \tag{25}$$

$$\mathbf{U}_{j+3}^* = \begin{Bmatrix} \operatorname{Re} Z_{4k} \\ \operatorname{Im} Z_{4k} \end{Bmatrix} \quad \text{with } Z_{4k} = -(1+\nu)\bar{z}^k \tag{26}$$

where $z = x+iy$, $\bar{z} = x-iy$ and $i = \sqrt{-1}$, Re(Z) and Im(Z) stand for the real part and the imaginary part of Z, respectively, and where U_j^* satisfies

$$\mathbf{L}_{in}\mathbf{U}_j^* = 0 \tag{27}$$

with

$$\mathbf{L}_{in} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}, \quad \mathbf{U}_j^* = \begin{Bmatrix} U_1^* \\ U_2^* \end{Bmatrix}_j \tag{28}$$

What follows is to derive the Trefftz functions of (16)–(18). Following Qin (1993), these three equations can be transformed into a convenient form

$$\nabla^2 f - \lambda^2 f = 0 \tag{29}$$

$$D\nabla^4 g + \frac{\bar{k}}{C} D\nabla^2 g - \bar{k} g + q = 0 \tag{30}$$

where f and g are two of Hu's functions (Qin, 1993), $\bar{k}$ is the subgrade reaction operator, $\bar{k} = k_w$ for Winkler-type foundation, $\bar{k} = k_p - G_p\nabla^2$ for Pasternak-type foundation, and

$$W = g - \frac{D}{C}\nabla^2 g \tag{31}$$

$$\varphi_x = g_{,x} + f_{,y} \tag{32}$$

$$\varphi_y = g_{,y} - f_{,x} \tag{33}$$

Equation (29) is the modified Helmholz equation, for which its Trefftz functions can be expressed, in polar coordinates r and θ, as

$$f_m = I_m(\lambda r)\cos m\theta, \quad f_{m+1} = I_m(\lambda r)\sin m\theta \; (m = 0, 1, 2, \ldots) \tag{34}$$

where $I_m()$ stands for the modified Bessel function of the first kind with order m.

Turning our attention to derive Trefftz functions for equation (30). To this end, consider the homogeneous equation

$$D\nabla^4 g + \frac{\bar{k}}{C} D\nabla^2 g - \bar{k} g = D(\nabla^2 + C_1)(\nabla^2 - C_2) g = 0 \tag{35}$$

where

$$C_1 = \sqrt{(k/2C)^2 + k/D} + k/2C$$
$$C_2 = \sqrt{(k/2C)^2 + k/D} - k/2C$$

for a Winkler-type foundation, or

$$C_1 = \frac{\sqrt{b} + k_p/C + G_p/D}{2(1 - G_p/C)}$$
$$C_2 = \frac{\sqrt{b} - k_p/C - G_p/D}{2(1 - G_p/C)}$$
$$b = (k_p/C + G_p/D)^2 + 4k_p(1 - G_p/C)/D$$

for a Pasternak-type foundation.

Following Qin (1993), we may set

$$(\nabla^2 + C_1)g = A \tag{36}$$

$$(\nabla^2 - C_2)g = B \tag{37}$$

The Trefftz functions for (36) and (37) can be expressed by

$$A_m = I_m(r\sqrt{C_2})\cos m\theta, \quad A_{m+1} = I_m(r\sqrt{C_2})\sin m\theta \tag{38}$$

$$B_m = J_m(r\sqrt{C_1})\cos m\theta, \quad B_{m+1} = J_m(r\sqrt{C_1})\sin m\theta \tag{39}$$

where $J_m()$ is Bessel function of the first kind with order m.

Therefore the Trefftz functions of (35) can be given from the following sequence

$$g(r,\theta) = c_0F_0(r) + \sum_{m=1}^{\infty} [c_mF_m(r)\cos m\theta + d_mF_m(r)\sin m\theta] \tag{40}$$

where

$$F_m(r) = I_m(r\sqrt{C_2}) - J_m(r\sqrt{C_1})$$

2.3. *Assumed fields*

The HT FE model is based on assuming two distinct displacements: the internal field **U** and the frame function $\tilde{\mathbf{U}}$ (Jirousek and Venkatesh, 1992). The field **U** fulfills identically the governing differential equations and is assumed as

$$\mathbf{u} = \begin{Bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{Bmatrix} = \begin{Bmatrix} \Delta\mathring{U}_1 \\ \Delta\mathring{U}_2 \end{Bmatrix} + \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} \mathbf{c}_{in} = \mathring{\mathbf{u}} + \mathbf{N}_{in}\mathbf{c}_{in} \tag{41}$$

$$\mathbf{w} = \begin{Bmatrix} \dot{W} \\ \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{Bmatrix} = \begin{Bmatrix} \Delta\mathring{W} \\ \Delta\mathring{\varphi}_1 \\ \delta\mathring{\varphi}_2 \end{Bmatrix} + \begin{Bmatrix} N_3 \\ N_4 \\ N_5 \end{Bmatrix} \mathbf{c}_{out} = \mathring{\mathbf{w}} + \mathbf{N}_{out}\mathbf{c}_{out} \tag{42}$$

where $\mathbf{c}_{in}$ and $\mathbf{c}_{out}$ are two undetermined vectors and $\mathring{\mathbf{u}}$, $\mathring{\mathbf{w}}$, $\mathbf{N}_{in}$, $\mathbf{N}_{out}$ are known functions which satisfy

$$\mathbf{L}_{in}\mathring{\mathbf{u}} = \begin{Bmatrix} \dot{P}_1 \\ \dot{P}_2 \end{Bmatrix}, \quad \mathbf{L}_{in}\begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} = 0\,(\text{on } \Omega_e)$$

$$\mathbf{L}_{out}\mathring{\mathbf{w}} = \begin{Bmatrix} \dot{P}_3 \\ 0 \\ 0 \end{Bmatrix}, \quad \mathbf{L}_{out}\begin{Bmatrix} N_3 \\ N_4 \\ N_5 \end{Bmatrix} = 0\,(\text{on } \Omega_e)$$

$$\mathbf{c} = \begin{Bmatrix} \mathbf{c}_{in} \\ \mathbf{c}_{out} \end{Bmatrix}, \quad \dot{\mathbf{U}} = \{\dot{U}_1\,\dot{U}_2\,\dot{W}\dot{\varphi}_1\dot{\varphi}_2\}^{\mathrm{T}}$$

$$\mathbf{L}_{out} = \begin{bmatrix} L_{33} & L_{34} & L_{35} \\ L_{43} & L_{44} & L_{45} \\ L_{53} & L_{54} & L_{55} \end{bmatrix}$$

and where $\mathbf{N}_{in}$ and $\mathbf{N}_{out}$ are formed by a suitably truncated complete system of (23)–(26), (34) and (40). While the particular solution $\mathring{\mathbf{w}}$ is given by

$$\mathring{\mathbf{w}} = \{\mathring{g} - D\nabla^2\mathring{g}/C, \mathring{g}_{,x}, \mathring{g}_{,y}\}^{\mathrm{T}} \tag{43}$$

such that

$$D\nabla^4\mathring{g} + \frac{\bar{k}}{C}D\nabla^2\mathring{g} - \bar{k}\mathring{g} = -q \tag{44}$$

and

$$(\mathbf{N}_{out})_k = \{g_i - D\nabla_i/C, g_{i,x} + f_{j,y}, g_{i,y} - f_{j,x}\}^{\mathrm{T}} \tag{45}$$

where

$$g_0 = 0, \quad g_1 = F_0(r), \quad g_2 = F_1(r)\cos\theta, \quad g_3 = F_1(r)\sin\theta, \ldots$$
$$f_0 = 0, \quad f_1 = I_0(\lambda r), \quad f_2 = I_1(\lambda r)\cos\theta, \quad f_3 = I_1(\lambda r)\sin\theta, \ldots$$

The indices i, j and k are ordered as shown in Table 1. Such ordering makes it easy to preserve the invariant properties of element under the rotation of its coordinate axis when the set is truncated.

Furthermore, to enforce on $\mathbf{U}$ the conformity, $\mathbf{U}^e = \mathbf{U}^f$ on $\partial\Omega_e \cap \partial\Omega_f$ (where "e" and "f" stand for any two neighboring elements), we will use an auxiliary interelement frame field $\tilde{\mathbf{U}}$ approximated in terms of the same degrees of freedom (DOF), $\mathbf{d}$, as in the conventional elements. The conforming frame field are assumed as

$$\tilde{\mathbf{U}} = \tilde{\mathbf{N}}\mathbf{d} \tag{46}$$

where

Table 1. Ordering of indexes in eqn (45)

k	1	2	3	4	5	6	7	8	...
i	1	0	2	3	0	4	5	0	...
j	0	1	0	0	2	0	0	3	...

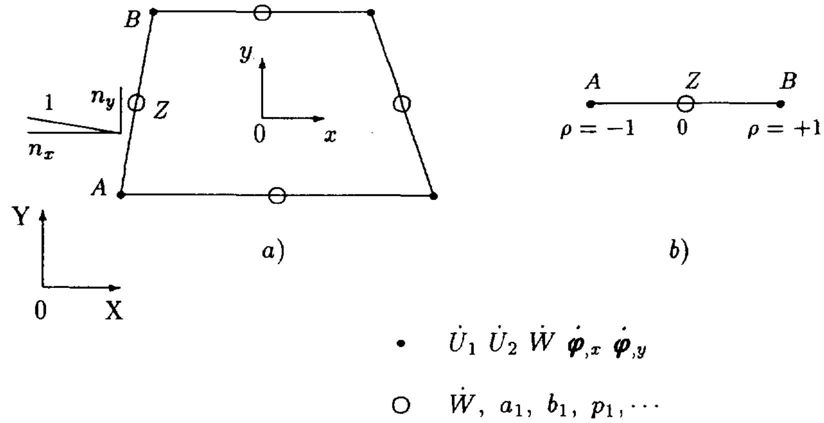

Fig. 1. The HT p-element.

$$\tilde{\mathbf{U}} = \begin{Bmatrix} \tilde{\mathbf{u}} \\ \tilde{\mathbf{w}} \end{Bmatrix} \tag{47}$$

$$\tilde{\mathbf{u}} = \{\tilde{U}_1 \tilde{U}_2\}^{\mathrm{T}} = \begin{bmatrix} \tilde{\mathbf{N}}_1 \\ \tilde{\mathbf{N}}_2 \end{bmatrix} \mathbf{d}_{in} \tag{48}$$

$$\tilde{\mathbf{w}} = \{\tilde{W} \tilde{\varphi}_1 \tilde{\varphi}_2\}^{\mathrm{T}} = \begin{bmatrix} \tilde{\mathbf{N}}_3 \\ \tilde{\mathbf{N}}_4 \\ \tilde{\mathbf{N}}_5 \end{bmatrix} \mathbf{d}_{out} \tag{49}$$

$$\mathbf{d} = \begin{Bmatrix} \mathbf{d}_{in} \\ \mathbf{d}_{out} \end{Bmatrix} \tag{50}$$

and where $\mathbf{d}_{in}$ and $\mathbf{d}_{out}$ stand for vectors of in-plane and out-of-plane displacement nodal parameters, and $\tilde{\mathbf{N}}_i$ $(i = 1, 2, 3, 4, 5)$ are the usual FE interpolation functions.

In the development of the present p-method elements, the following assumptions are adopted. First of all, the element may be of a general quadrilateral shape or a triangular shape with five DOF $(\dot{U}_1, \dot{U}_2, \dot{W}, \dot{\varphi}_1, \dot{\varphi}_2)$ at each corner node and with one DOF $(\dot{W})$, at each mid-side node (see Fig. 1). In this case, the in-plane displacements and rotations are linear along a side of the element boundary, and the boundary deflection is quadratic. Secondly to achieve higher order variations, an optional number of extra hierarchic modes is introduced along with the hierarchic DOF, a_i for $\tilde{U}_1$, b_i for $\tilde{U}_2$, p_i for $\tilde{W}$, q_i for $\tilde{\varphi}_1$, r_i for $\tilde{\varphi}_2$, which are conveniently associated with mid-side node Z (see Fig. 1). Thus, along the side A-Z-B of a particular element (see Fig. 1), the frame functions are finally defined by

$$\tilde{U}_1 = \tilde{N}_1 \dot{U}_{1A} + \tilde{N}_2 \dot{U}_{1B} + \sum_i \gamma^{i-1} M_i a_i \tag{51}$$

$$\tilde{U}_2 = \tilde{N}_1 \dot{U}_{2A} + \tilde{N}_2 \dot{U}_{2B} + \sum_i \gamma^{i-1} M_i b_i \tag{52}$$

$$\tilde{W} = \tilde{N}_3 \dot{W}_A + \tilde{N}_4 \dot{W}_B + \tilde{N}_5 \dot{W}_Z + \sum_i \gamma^i \rho M_i p_i \tag{53}$$

$$\tilde{\varphi}_1 = \tilde{N}_1 \dot{\varphi}_{1A} + \tilde{N}_2 \dot{\varphi}_{1B} + \sum_i \gamma^{i-1} M_i q_i \tag{54}$$

$$\tilde{\varphi}_2 = \tilde{N}_1\dot{\varphi}_{2A}+\tilde{N}_2\dot{\varphi}_{2B}+\sum_i \gamma^{i-1}M_i r_i \tag{55}$$

where ρ is shown in Fig. 1 and where

$$\tilde{N}_1 = (1-\rho)/2,\quad \tilde{N}_2 = (1+\rho)/2,\quad \tilde{N}_3 = \rho(\rho-1)/2,\quad \tilde{N}_4 = \rho(\rho+1)/2,$$
$$\tilde{N}_5 = 1-\rho^2,\quad M_i = \rho^{i-1}(1-\rho^2)$$

The coefficient γ is equal to $+1$ or -1 according to the orientation of the side A-Z-B (see Fig. 1) in the global coordinate system (X, Y):

$$\gamma = \begin{cases} +1 & \text{if } X_B-X_A \leqslant Y_B-Y_A \\ -1 & \text{if } X_B-X_A > Y_B-Y_A \end{cases} \tag{56}$$

The purpose of the coefficient γ is to ensure a univocal definition of the frame functions $\tilde{U}$ in terms of parameters $a_i b_i p_i q_i$ and r_i, common to two elements sharing the mid-side node Z.

The generalized boundary forces and displacements can be easily derived from (19)–(22), (41), (42) and (46), and denote

$$\mathbf{v} = \begin{Bmatrix} \dot{U}_n \\ \dot{U}_s \end{Bmatrix} = \begin{bmatrix} n_1 & n_2 \\ s_1 & s_2 \end{bmatrix} \begin{Bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{Bmatrix} = \mathring{\mathbf{v}}+\mathbf{Q}_1\mathbf{c}_{in} \tag{57}$$

$$\mathbf{F} = \begin{Bmatrix} \dot{N}_n \\ \dot{N}_{ns} \end{Bmatrix} = \begin{bmatrix} n_1^2 & n_2^2 & 2n_1n_2 \\ n_1s_1 & n_2s_2 & n_1s_2+n_2s_1 \end{bmatrix} \begin{Bmatrix} \dot{N}_X \\ \dot{N}_Y \\ \dot{N}_{xy} \end{Bmatrix} = \mathring{\mathbf{N}}+\mathbf{Q}_2\mathbf{c}_{in} \tag{58}$$

$$\mathbf{w}_b = \begin{Bmatrix} \dot{W} \\ \dot{\varphi}_n \\ \dot{\varphi}_s \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & n_1 & n_2 \\ 0 & s_1 & s_2 \end{bmatrix} \begin{Bmatrix} \dot{W} \\ \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{Bmatrix} = \mathring{\mathbf{w}}_b+\mathbf{Q}_3\mathbf{c}_{out} \tag{59}$$

$$\mathbf{M} = \begin{Bmatrix} \dot{R} \\ \dot{M}_n \\ \dot{M}_{ns} \end{Bmatrix} = \mathbf{AG} = \mathring{\mathbf{M}}+\mathbf{Q}_4\mathbf{d}_{in} \tag{60}$$

$$\tilde{\mathbf{v}} = \begin{Bmatrix} \tilde{U}_n \\ \tilde{U}_s \end{Bmatrix} = \begin{bmatrix} n_1 & n_2 \\ s_1 & s_2 \end{bmatrix} \begin{Bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \end{Bmatrix} = \mathbf{Q}_5\mathbf{d}_{in} \tag{61}$$

$$\tilde{\mathbf{w}}_b = \begin{Bmatrix} \tilde{W} \\ \tilde{\varphi}_n \\ \tilde{\varphi}_s \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & n_1 & n_2 \\ 0 & s_1 & s_2 \end{bmatrix} \begin{Bmatrix} \tilde{W} \\ \tilde{\varphi}_1 \\ \tilde{\varphi}_2 \end{Bmatrix} = \mathbf{Q}_6\mathbf{d}_{out} \tag{62}$$

where

$$\mathbf{Q}_i = \begin{bmatrix} \mathbf{Q}_{i1} \\ \mathbf{Q}_{i2} \end{bmatrix},\quad (i = 1,2,5) \qquad \mathbf{Q}_i = \begin{bmatrix} \mathbf{Q}_{i1} \\ \mathbf{Q}_{i2} \\ \mathbf{Q}_{i3} \end{bmatrix},\quad (i = 3,4,6)$$

$$\mathring{\mathbf{v}} = \{\Delta\mathring{U}_n \Delta\mathring{U}_s\}^{\mathrm{T}},$$

$$\mathring{\mathbf{N}} = \{\Delta\mathring{N}_n \Delta\mathring{N}_{ns}\}^{\mathrm{T}},$$

$$\mathring{\mathbf{w}}_b = \{\Delta\mathring{W}\Delta\mathring{\varphi}_n\Delta\mathring{\varphi}_s\}^{\mathrm{T}},$$

$$\mathring{\mathbf{M}} = \{\Delta\mathring{R}\Delta\mathring{M}_n\Delta\mathring{M}_{ns}\}^{\mathrm{T}},$$

$$\mathbf{A} = \begin{bmatrix} n_x & n_y & 0 & 0 & 0 \\ 0 & 0 & n_1^2 & n_2^2 & 2n_1n_2 \\ 0 & 0 & n_1s_1 & n_2s_2 & n_1s_2+n_2s_1 \end{bmatrix},$$

$$\mathbf{G} = \{\dot{Q}_x \quad \dot{Q}_y \quad \dot{M}_x \quad \dot{M}_y \quad \dot{M}_{xy}\}^{\mathrm{T}}$$

2.4. *Particular solution*

The particular solution of $\mathring{\mathbf{u}}$ and $\mathring{\mathbf{w}}$ can be obtained by means of their source (or Green's) functions. The source functions of (14)–(18) used are as follows (Qin, 1993)

$$U^*_{ij}(p,q) = \frac{1+v}{4\pi E}[-(3-v)Lnr_{pq}\delta_{ij}+(1+v)r_{pq,i}r_{pq,j}],$$

$$U^*_{33}(p,q) = AC_2K_0(Z_2)(1-DC_2/C)+BY_0(Z_1)(1+DC_1/C),$$

$$U^*_{43}(p,q) = -[B\sqrt{C_1}Y_1(Z_1)+A\sqrt{C_2}K_1(Z_2)]\cos(\beta-\phi)$$

$$U^*_{53}(p,q) = -[B\sqrt{C_1}Y_1(Z_1)+A\sqrt{C_2}K_1(Z_2)]\sin(\beta-\phi)$$

$$r_{pq} = \sqrt{(x_q-x_p)^2+(y_q-y_p)^2}$$

where $U^*_{mn}(p,q)$ designates the in-plane displacements (for $m = 1,2$) or deflection (for $m = 3$) or rotations (for $m = 4,5$) at field point q of an infinite plate when a unit point force (for $n = 1,2,3$) is applied at the source point p, $K_m()$ is a modified Bessel function of the second kind with order m, $Y_m()$ is a Bessel function of the second kind with order m, angles β and ϕ are shown in Fig. 2, and where

$$A = \frac{1}{2\pi D(C_1+C_2)}, \quad B = -\frac{1}{4D(C_1+C_2)}, \quad Z_1 = r_{pq}\sqrt{C_1}, \quad Z_2 = r_{pq}\sqrt{C_2},$$

Thus the particular solutions of (41) and (42) can be expressed as

$$\mathring{\mathbf{u}} = \begin{Bmatrix}\Delta\mathring{U}_1 \\ \Delta\mathring{U}_2\end{Bmatrix} = \iint_\Omega \dot{P}_j \begin{Bmatrix} U^*_{1j} \\ U^*_{2j}\end{Bmatrix} \mathrm{d}\Omega \tag{63}$$

$$\mathring{\mathbf{w}} = \begin{Bmatrix}\Delta\mathring{W} \\ \Delta\mathring{\varphi}_1 \\ \Delta\mathring{\varphi}_2\end{Bmatrix} = \iint_\Omega \dot{P}_3 \begin{Bmatrix} U^*_{33} \\ U^*_{43} \\ U^*_{53}\end{Bmatrix} \mathrm{d}\Omega \tag{64}$$

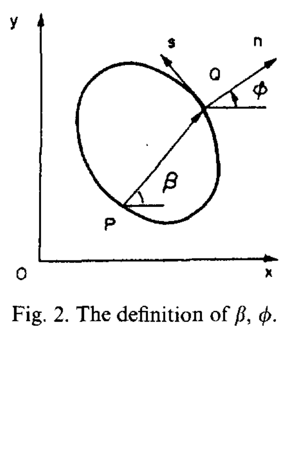

Fig. 2. The definition of β, ϕ.

The area integration in (63) and (64) will be performed by numerical quadrature using the Gauss-Legendre rule.

2.5. *Modified variational principle*

The HT FE formulation for nonlinear analysis of thick plates can be obtained by means of a modified principle (Qin, 1994). The related functional used for deriving HT FE formulation can be given in the form (Qin, 1994)

$$\Gamma_{in}^{m} = \sum_{e}\left\{\Gamma_{in}^{e} + \int_{\partial\Omega_e^2} (\dot{\bar{N}}_n^* - \dot{N}_n)\dot{U}_n \mathrm{d}c + \int_{\partial\Omega_e^4} (\dot{\bar{N}}_{ns}^* - \dot{N}_{ns})\dot{U}_s \,\mathrm{d}c - \int_{\partial\Omega_e^{11}} \mathbf{F}^{\mathrm{T}}\tilde{\mathbf{v}}\,\mathrm{d}c\right\} \tag{65}$$

$$\Gamma_{out}^{m} = \sum_{e}\left\{\Gamma_{out}^{e} + \int_{\partial\Omega_e^6} (\dot{\bar{R}}^* - \dot{R})\dot{W}\,\mathrm{d}c + \int_{\partial\Omega_e^8} (\Delta\bar{M}_n - \dot{M}_n)\dot{\varphi}_n \,\mathrm{d}c \right.$$
$$\left. + \int_{\partial\Omega_e^{10}} (\Delta\bar{M}_{ns} - \dot{M}_{ns})\dot{\varphi}_s \,\mathrm{d}c - \int_{\partial\Omega_e^{11}} \mathbf{M}^{\mathrm{T}}\tilde{\mathbf{w}}_b \,\mathrm{d}c\right\} \tag{66}$$

where

$$\Delta X = \dot{X}$$

$$\Gamma_{in}^{e} = \iint_{\Omega_e} \dot{V}_{in}\,\mathrm{d}\Omega - \int_{\partial\Omega_e^1} \dot{N}_n \Delta\bar{U}_n \,\mathrm{d}c - \int_{\partial\Omega_e^3} \dot{N}_{ns}\Delta\bar{U}_s \,\mathrm{d}c$$

$$\Gamma_{out}^{e} = \iint_{\Omega_e} \dot{V}_{out}\,\mathrm{d}\Omega - \int_{\partial\Omega_e^5} \dot{R}\Delta\bar{W}\,\mathrm{d}c - \int_{\partial\Omega_e^7} \dot{M}_n \Delta\bar{\varphi}_n \,\mathrm{d}c - \int_{\partial\Omega_e^9} \dot{M}_{ns}\Delta\bar{\varphi}_s \,\mathrm{d}c$$

$$\dot{V}_{in} = \frac{1-2\nu}{6Eh}\dot{N}_{kl}\dot{N}_{kl} + \frac{1+\nu}{2Eh}\dot{N}_{ij}^*\dot{N}_{ij}^*$$

$$\dot{V}_{out} = \frac{1}{2D(1-\nu^2)}[(\dot{M}_x + \dot{M}_y)^2 + 2(1+\nu)(\dot{M}_{xy}^2 - \dot{M}_x\dot{M}_y)] + \frac{1}{2C}(\dot{Q}_x^2 + \dot{Q}_y^2) + V^*$$

$$\dot{N}_{ij}^* = \dot{N}_{ij} - \tfrac{1}{3}\dot{N}_{kk}\delta_{ij}$$

$$V^* = \begin{cases} k_w\dot{W}^2/2 & \text{for Winkler type foundation,} \\ (k_p\dot{W}^2 + G_p\dot{W}_{,i}\dot{W}_{,i})/2 & \text{for Pasternak type foundation} \end{cases}$$

and where (14)–(18) are assumed to be satisfied, *a priori*. The terminology "modified principle" refers, here, to the use of conventional potential functional Γ_{in} (or Γ_{out}) and some modified terms for the construction of a special variational principle.

The boundary $\partial\Omega$ of a particular element consists of the following parts

$$\partial\Omega_e = \partial\Omega_e^1 \cup \partial\Omega_e^2 \cup \partial\Omega_e^{11} = \partial\Omega_e^3 \cup \partial\Omega_e^4 \cup \partial\Omega_e^{11} = \partial\Omega_e^5 \cup \partial\Omega_e^6 \cup \partial\Omega_e^{11}$$
$$= \partial\Omega_e^7 \cup \partial\Omega_e^8 \cup \partial\Omega_e^{11} = \partial\Omega_e^9 \cup \partial\Omega_e^{10} \cup \partial\Omega_e^{11}$$

where

$$\partial\Omega_e^1 \subset C_{U_n} \cap \partial\Omega_e, \quad \partial\Omega_e^2 \subset C_{N_n} \cap \partial\Omega_e, \quad \partial\Omega_e^3 \subset C_{U_s} \cap \partial\Omega_e, \quad \partial\Omega_e^4 \subset C_{N_{ns}} \cap \partial\Omega_e,$$

$$\partial\Omega_e^5 \subset C_W \cap \partial\Omega_e, \quad \partial\Omega_e^6 \subset C_R \cap \partial\Omega_e, \quad \partial\Omega_e^7 \subset C_{\varphi_n} \cap \partial\Omega_e, \quad \partial\Omega_e^8 \subset C_{M_n} \cap \partial\Omega_e,$$

$$\partial\Omega_e^9 \subset C_{\varphi_s} \cap \partial\Omega_e, \quad \partial\Omega_e^{10} \subset C_{M_{ns}} \cap \partial\Omega_e,$$

and $\partial\Omega_e^{11}$ is the interelement boundary of the element. Consequently, we will discuss some properties and their proof on these two functionals. They are

(i) Modified complementary principle

$$\delta\Gamma_{in}^{m}=0 \Rightarrow (19),(21) \quad \text{and} \quad \dot{U}_n^e=\dot{U}_n^f, \quad \dot{U}_s^e=\dot{U}_s^f, \quad (\text{on } \partial\Omega_e\cap\partial\Omega_f) \tag{67}$$

$$\delta\Gamma_{out}^{m}=0 \Rightarrow (20),(22) \quad \text{and} \quad \dot{W}^e=\dot{W}^f, \quad \dot{\varphi}_n^e=\dot{\varphi}_n^f, \quad \dot{\varphi}_s^e=\dot{\varphi}_s^f, \quad (\text{on } \partial\Omega_e\cap\partial\Omega_f) \tag{68}$$

(ii) Theorems on the existence of extremum:

(a) if the expression

$$\iint_\Omega \delta^2\dot{V}_{in}\,\mathrm{d}\Omega-\int_{C_{N_n}}\delta\dot{N}_n\delta\dot{U}_n\,\mathrm{d}c-\int_{C_{N_{ns}}}\delta\dot{N}_{ns}\delta\dot{U}_s\,\mathrm{d}c-\sum_e\int_{\partial\Omega_e^{11}}\delta\mathbf{F}^{\mathrm{T}}\delta\tilde{\mathbf{v}}\,\mathrm{d}c \tag{69}$$

is uniformly positive (or negative) at the neighborhood of $\mathbf{u}_0(\mathbf{u}_0=\{\dot{U}_{10}\dot{U}_{20}\})$, where $\mathbf{u}_0$ is such a value that $\Gamma_{in}^m(\mathbf{u}_0)=(\Gamma_{in}^m)_0$, and where $(\Gamma_{in}^m)_0$ stands for the stationary value of Γ_{in}^m. We have

$$\Gamma_{in}^m \geqslant (\Gamma_{in}^m)_0 (\text{or } \Gamma_{in}^m \leqslant (\Gamma_{in}^m)_0) \tag{70}$$

(b) if the expression

$$\iint_\Omega \delta^2\dot{V}_{out}\,\mathrm{d}\Omega-\int_{C_R}\delta\dot{R}\delta\dot{W}\,\mathrm{d}c-\int_{C_{M_n}}\delta\dot{M}_n\delta\dot{\varphi}_n\,\mathrm{d}c-\int_{C_{M_n}}\delta\dot{M}_{ns}\delta\dot{\varphi}_s\,\mathrm{d}c-\sum_e\int_{\partial\Omega_e^{11}}\delta\mathbf{M}^{\mathrm{T}}\delta\tilde{\dot{W}}_b\,\mathrm{d}c \tag{71}$$

is uniformly positive (or negative) at the neighborhood of $\mathbf{w}_0(\mathbf{w}_0=\{\dot{W}_0\dot{\varphi}_{10}\dot{\varphi}_{20}\})$, where $\mathbf{w}_0$ is such a value that $\Gamma_{out}^m(\mathbf{w}_0)=(\Gamma_{out}^m)_0$, and where $(\Gamma_{out}^m)_0$ stands for the stationary value of Γ_{out}^m. We have

$$\Gamma_{out}^m \geqslant (\Gamma_{out}^m)_0 (\text{or } \Gamma_{out}^m \leqslant (\Gamma_{out}^m)_0) \tag{72}$$

where "e" and "f" stand for any two neighboring elements and where $\tilde{\mathbf{U}}^e=\tilde{\mathbf{U}}^f$ is identical on $\partial\Omega_e\cap\partial\Omega_f$ due to the property of the assumed frame field [see (46)].

PROOF: from the first, we derive the stationary conditions of functional Γ_{in}^m. To this end, taking variation of Γ_{in}^m and noting that (14) and (15) hold *a priori* by the previous assumption, one obtains

$$\begin{aligned}\delta\Gamma_{in}^m \overset{(14)(15)}{=\!=\!=} &\int_{C_{U_n}}(\dot{U}_n-\Delta\bar{U}_n)\delta\dot{N}_n\,\mathrm{d}c+\int_{C_{U_s}}(\dot{U}_s-\Delta\bar{U}_s)\delta\dot{N}_{ns}\,\mathrm{d}c-\int_{C_{N_n}}(\dot{N}_n-\Delta\bar{N}_n^*)\delta\dot{U}_n\,\mathrm{d}c\\ &-\int_{C_{N_{ns}}}(\dot{N}_{ns}-\Delta\bar{N}_{ns}^*)\delta\dot{U}_s\,\mathrm{d}c-\sum_e\int_{\partial\Omega_e^{11}}[(\dot{U}_n-\tilde{U}_n)\delta\dot{N}_n+(\dot{U}_s-\tilde{U}_s)\delta\dot{N}_{ns}]\,\mathrm{d}c\end{aligned} \tag{73}$$

where the constrained equality $\overset{(14)(15)}{=\!=\!=}$ stands for the eqns (14) and (15) being satisfied *a priori*. Therefore, the Euler equations for (73) are (19), (21) and

$$\dot{U}_n^e=\dot{U}_n^f, \quad \dot{U}_s^e=\dot{U}_s^f, \quad (\text{on } \partial\Omega_e\cap\partial\Omega_f)$$

The principle (67) has been, thus, proved.

As for the proof of the theorem on the existence of extremum, we may complete it by way of the so-called second variational approach (Simpson and Spector, 1987; He and Qin, 1993). In doing this, taking variation of $\delta\,\Gamma^m_{in}$ and using the constrained conditions (14) and (15), we see

$$\delta^2\Gamma^m_{in} = \iint_{\Omega} \delta^2 \dot{V}_{in}\,\mathrm{d}\Omega - \int_{C_{N_n}} \delta\dot{N}_n \delta\dot{U}_n\,\mathrm{d}c - \int_{C_{N_{ns}}} \delta\dot{N}_{ns}\delta\dot{U}_s\,\mathrm{d}c - \sum_e \int_{\Omega_e^{11}} \delta\mathbf{F}^{\mathrm{T}}\delta\tilde{\mathbf{v}}\,\mathrm{d}c$$

$$= \text{expression (69)} \quad (74)$$

So the theorem has been proved from the sufficient condition on the existence of local extreme of a functional (Simpson and Spector, 1987). With the same way as above, one may easily prove the inequality (72), and we omit those details here.

2.6. *Element matrix*

Element stiffness matrices may be obtained by setting $\delta(\Gamma^m_{in})_e = 0$ and $\delta(\Gamma^m_{out})_e = 0$. To simplify the derivation, all domain integrals in (65) and (66) are transformed into boundary ones except those loading terms by use of solution properties of the intraelement trial functions, for which the functionals $(\Gamma^m_{out})_e$ and $(\Gamma^m_{in})_e$ are rewritten as

$$(\Gamma^m_{in})_e = \frac{1}{2}\iint_{\Omega_e} \dot{P}_i\dot{U}_i\,\mathrm{d}\Omega - \int_{\partial\Omega_e^1} \dot{N}_n\Delta\bar{U}_n\,\mathrm{d}c - \int_{\partial\Omega_e^3} \dot{N}_{ns}\Delta\bar{U}_s\,\mathrm{d}c$$
$$- \int_{\partial\Omega_e^2} (\dot{N}_n - \bar{N}_n^*)\dot{U}_n\,\mathrm{d}c - \int_{\partial\Omega_e^4} (\dot{N}_{ns} - \bar{N}_{ns}^*)\dot{U}_s\,\mathrm{d}c + \frac{1}{2}\int_{\partial\Omega_e} \mathbf{F}^{\mathrm{T}}\mathbf{v}\,\mathrm{d}c - \int_{\partial\Omega_e^{11}} \mathbf{F}^{\mathrm{T}}\tilde{\mathbf{v}}\,\mathrm{d}c \quad (75)$$

$$(\Gamma^m_{out})_e = \frac{1}{2}\iint_{\Omega_e} \dot{P}_3\dot{W}\,\mathrm{d}\Omega - \int_{\partial\Omega_e^5} \dot{R}\Delta\bar{W}\,\mathrm{d}c - \int_{\partial\Omega_e^7} \dot{M}_n\Delta\bar{\varphi}_n\,\mathrm{d}c - \int_{\partial\Omega_e^9} \dot{M}_{ns}\Delta\bar{\varphi}_s\,\mathrm{d}c$$
$$- \int_{\partial\Omega_e^6} (\dot{R} - \bar{R}^*)\dot{W}\,\mathrm{d}c - \int_{\partial\Omega_e^8} (\dot{M}_n - \Delta\bar{M}_n)\dot{\varphi}_n\,\mathrm{d}c - \int_{\partial\Omega_e^{10}} (\dot{M}_{ns} - \Delta\bar{M}_{ns})\dot{\varphi}_s\,\mathrm{d}c$$
$$+ \frac{1}{2}\int_{\partial\Omega_e} \mathbf{M}^{\mathrm{T}}\mathbf{w}_b\,\mathrm{d}c - \int_{\partial\Omega_e^{11}} \mathbf{M}^{\mathrm{T}}\mathbf{w}_b\,\mathrm{d}c \quad (76)$$

The substitution of (41), (42), (46) and (57)–(62) into (75) and (76), obtains

$$(\Gamma^m_{in})_e = -\mathbf{c}_{in}^{\mathrm{T}}\mathbf{H}_{in}\mathbf{c}_{in}/2 + \mathbf{c}_{in}^{\mathrm{T}}\mathbf{S}_{in}\mathbf{d}_{in} + \mathbf{c}_{in}^{\mathrm{T}}r_1 + \mathbf{d}_{in}^{\mathrm{T}}r_2 + \text{terms without } \mathbf{c}_{in} \text{ or } \mathbf{d}_{in} \quad (77)$$

$$(\Gamma^m_{out})_e = -\mathbf{c}_{out}^{\mathrm{T}}\mathbf{H}_{out}\mathbf{c}_{out}/2 + \mathbf{c}_{out}^{\mathrm{T}}\mathbf{S}_{out}\mathbf{d}_{out} + \mathbf{c}_{out}^{\mathrm{T}}r_3 + \mathbf{d}_{out}^{\mathrm{T}}r_4 + \text{terms without } \mathbf{c}_{out} \text{ or } \mathbf{d}_{out} \quad (78)$$

where

$$\mathbf{H}_{in} = \mathbf{H}_{in}^* + (\mathbf{H}_{in}^*)^{\mathrm{T}}$$
$$\mathbf{H}_{in}^* = -\frac{1}{2}\int_{\partial\Omega_e} \mathbf{Q}_2^{\mathrm{T}}\mathbf{Q}_1\,\mathrm{d}c + \int_{\partial\Omega_e^2} \mathbf{Q}_{21}^{\mathrm{T}}\mathbf{Q}_{11}\,\mathrm{d}c + \int_{\partial\Omega_e^4} \mathbf{Q}_{22}^{\mathrm{T}}\mathbf{Q}_{12}\,\mathrm{d}c \quad (79)$$

$$\mathbf{S}_{in} = -\int_{\partial\Omega_e^{11}} \mathbf{Q}_2^{\mathrm{T}}\mathbf{Q}_5\,\mathrm{d}c \quad (80)$$

$$\mathbf{r}_1 = \frac{1}{2}\iint_{\Omega_e} (\mathbf{N}_1^{\mathrm{T}}\dot{\mathbf{P}}_1 + \mathbf{N}_2^{\mathrm{T}}\dot{\mathbf{P}}_2)\,\mathrm{d}c + \frac{1}{2}\int_{\partial\Omega_e} (\mathbf{Q}_1^{\mathrm{T}}\mathring{\mathbf{N}} + \mathbf{Q}_2^{\mathrm{T}}\mathring{\mathbf{u}})\,\mathrm{d}c$$
$$- \int_{\partial\Omega_e^1} \Delta\bar{U}_n\mathbf{Q}_{21}^{\mathrm{T}}\,\mathrm{d}c - \int_{\partial\Omega_e^3} \Delta\bar{U}_s\mathbf{Q}_{22}^{\mathrm{T}}\,\mathrm{d}c + \int_{\partial\Omega_e^2} \bar{N}_n^*\mathbf{Q}_{11}^{\mathrm{T}}\,\mathrm{d}c$$
$$+ \int_{\partial\Omega_e^4} \bar{N}_{ns}^*\mathbf{Q}_{12}^{\mathrm{T}}\,\mathrm{d}c - \int_{\partial\Omega_e^2} \Delta\mathring{N}_n\mathbf{Q}_{51}^{\mathrm{T}}\,\mathrm{d}c - \int_{\partial\Omega_e^4} \Delta\mathring{N}_{ns}\mathbf{Q}_{52}^{\mathrm{T}}\,\mathrm{d}c$$
$$- \int_{\partial\Omega_e^1} \Delta\mathring{U}_n\mathbf{Q}_{21}^{\mathrm{T}}\,\mathrm{d}c - \int_{\partial\Omega_e^3} \Delta\mathring{U}_s\mathbf{Q}_{22}^{\mathrm{T}}\,\mathrm{d}c \tag{81}$$

$$\mathbf{r}_2 = \int_{\partial\Omega_e^{11}} \mathbf{Q}_5^{\mathrm{T}}\mathring{\mathbf{N}}\,\mathrm{d}c \tag{82}$$

$$\mathbf{H}_{out} = \mathbf{H}_{out}^* + (\mathbf{H}_{out}^*)^{\mathrm{T}} \tag{83}$$

$$\mathbf{H}_{out}^* = -\frac{1}{2}\int_{\partial\Omega_e} \mathbf{Q}_4^{\mathrm{T}}\mathbf{Q}_3\,\mathrm{d}c + \int_{\partial\Omega_e^6} \mathbf{Q}_{41}^{\mathrm{T}}\mathbf{Q}_{31}\,\mathrm{d}c + \int_{\partial\Omega_e^8} \mathbf{Q}_{42}^{\mathrm{T}}\mathbf{Q}_{32}\,\mathrm{d}c + \int_{\partial\Omega_e^{10}} \mathbf{Q}_{43}^{\mathrm{T}}\mathbf{Q}_{33}\,\mathrm{d}c$$
$$\mathbf{S}_{out} = -\int_{\partial\Omega_e^{11}} \mathbf{Q}_4^{\mathrm{T}}\mathbf{Q}_6\,\mathrm{d}c \tag{84}$$

$$\mathbf{r}_3 = \frac{1}{2}\iint_{\Omega_e} \mathbf{N}_3^{\mathrm{T}}\dot{P}_3\,\mathrm{d}c + \frac{1}{2}\int_{\partial\Omega_e} (\mathbf{Q}_3^{\mathrm{T}}\mathring{\mathbf{M}} + \mathbf{Q}_4^{\mathrm{T}}\mathring{\mathbf{w}}_b)\,\mathrm{d}c$$
$$- \int_{\partial\Omega_e^5} \Delta\bar{W}\mathbf{Q}_{41}^{\mathrm{T}}\,\mathrm{d}c - \int_{\partial\Omega_e^7} \Delta\bar{\varphi}_n\mathbf{Q}_{42}^{\mathrm{T}}\,\mathrm{d}c - \int_{\partial\Omega_e^9} \Delta\bar{\varphi}_s\mathbf{Q}_{43}^{\mathrm{T}}\,\mathrm{d}c$$
$$- \int_{\partial\Omega_e^6} [\Delta\mathring{W}\mathbf{Q}_{41}^{\mathrm{T}} + \mathbf{Q}_{31}^{\mathrm{T}}(\delta\mathring{R} - \bar{R}^*)]\,\mathrm{d}c - \int_{\partial\Omega_e^8} [\Delta\mathring{\varphi}_n\mathbf{Q}_{42}^{\mathrm{T}} + \mathbf{Q}_{32}^{\mathrm{T}}(\Delta\mathring{M}_n - \Delta\bar{\mathbf{M}}_n)]\,\mathrm{d}c$$
$$- \int_{\partial\Omega_e^{10}} [\Delta\mathring{\varphi}_s\mathbf{Q}_{43}^{\mathrm{T}} + \mathbf{Q}_{33}^{\mathrm{T}}(\Delta\mathring{M}_{ns} - \Delta\bar{M}_{ns})]\,\mathrm{d}c \tag{85}$$

$$\mathbf{r}_4 = -\int_{\partial\Omega_e^{11}} \mathbf{Q}_6^{\mathrm{T}}\mathring{\mathbf{M}}\,\mathrm{d}c \tag{86}$$

Note that all terms not involving $\mathbf{c}$ and $\mathbf{d}$ are of no significance for an approximate solution and are therefore not listed explicitly.

To obtain the element stiffness matrices, taking vanishing variation of (77) and (78) with respect to $\mathbf{c}$ at the element level, we have

$$\frac{\partial(\Gamma_{in}^m)_e}{\partial\mathbf{c}_{in}^{\mathrm{T}}} = -\mathbf{H}_{in}\mathbf{c}_{in} + \mathbf{S}_{in}\mathbf{d}_{in} + \mathbf{r}_1 \tag{87}$$

$$\frac{\partial(\Gamma_{out}^m)_e}{\partial\mathbf{c}_{out}^{\mathrm{T}}} = -\mathbf{H}_{out}\mathbf{c}_{out} + \mathbf{S}_{out}\mathbf{d}_{out} + \mathbf{r}_3 \tag{88}$$

which lead to

$$\mathbf{c}_{in} = \mathbf{G}_{in}\mathbf{d}_{in} + \mathbf{g}_{in} \tag{89}$$

$$\mathbf{c}_{out} = \mathbf{G}_{out}\mathbf{d}_{out} + \mathbf{g}_{out} \tag{90}$$

where

$$\mathbf{G}_{in} = \mathbf{H}_{in}^{-1}\mathbf{S}_{in}, \quad \mathbf{g}_{in} = \mathbf{H}_{in}^{-1}\mathbf{r}_1, \tag{91}$$

$$\mathbf{G}_{out} = \mathbf{H}_{out}^{-1}\mathbf{S}_{out}, \quad \mathbf{g}_{out} = \mathbf{H}_{out}^{-1}\mathbf{r}_3 \tag{92}$$

As a consequence the functionals $(\Gamma_{in}^m)_e$ and $(\Gamma_{out}^m)_e$ can be expressed only in terms of $\mathbf{d}$ and other known matrices.

$$(\Gamma_{in}^m)_e = -\mathbf{d}_{in}^{\mathrm{T}}\mathbf{G}_{in}^{\mathrm{T}}\mathbf{H}_{in}\mathbf{G}_{in}\mathbf{d}_{in}/2 + \mathbf{d}_{in}^{\mathrm{T}}[\mathbf{G}_{in}^{\mathrm{T}}\mathbf{H}_{in}\mathbf{g}_{in} + \mathbf{r}_2] + \text{terms without } \mathbf{d}_{in} \tag{93}$$

$$(\Gamma_{out}^m)_e = -\mathbf{d}_{out}^{\mathrm{T}}\mathbf{G}_{out}^{\mathrm{T}}\mathbf{H}_{out}\mathbf{G}_{out}\mathbf{d}_{out}/2 + \mathbf{d}_{out}^{\mathrm{T}}[\mathbf{G}_{out}^{\mathrm{T}}\mathbf{H}_{out}\mathbf{g}_{out} + \mathbf{r}_4] + \text{terms without } \mathbf{d}_{out} \tag{94}$$

and then the customary force-displacement relationships may be given by

$$\mathbf{K}_{in}\mathbf{d}_{in} = \mathbf{P}_{in} \tag{95}$$

$$\mathbf{K}_{out}\mathbf{d}_{out} = \mathbf{P}_{out} \tag{96}$$

where

$$\mathbf{K}_{in} = \mathbf{G}_{in}^{\mathrm{T}}\mathbf{H}_{in}\mathbf{G}_{in} \tag{97}$$

$$\mathbf{K}_{out} = \mathbf{G}_{out}^{\mathrm{T}}\mathbf{H}_{out}\mathbf{G}_{out} \tag{98}$$

$$\mathbf{P}_{in} = \mathbf{G}_{in}^{\mathrm{T}}\mathbf{H}_{in}\mathbf{g}_{in} + \mathbf{r}_2 \tag{99}$$

$$\mathbf{P}_{out} = \mathbf{G}_{out}^{\mathrm{T}}\mathbf{H}_{out}\mathbf{g}_{out} + \mathbf{r}_4 \tag{100}$$

Moreover, to ensure a good numerical conditioning during the inversion of matrices $\mathbf{H}_{in}$ and $\mathbf{H}_{out}$ the homogeneous solutions $\mathbf{N}_k$ in (41) and (42) have to be expressed in terms of suitably scaled local coordinates (x, y) originated at the element centroid (Fig. 1)

$$x = (X - X_c)/a, \quad y = (Y - Y_c)/a$$

where X and Y are global coordinates, X_c and Y_c stand for global coordinates at centroid of the element, and a is the average distance between the centroid and node i of the element:

$$a = \sum_i^N (x_i^2 + y_i^2)^{1/2}/N$$

and where N is the nodal number of the element.

Besides, it should be pointed out that the standard HT formulation implies that all terms representing the rigid body modes (three in each case, in-plane or out-of-plane) have been discarded from the intraelement field to prevent the matrices $\mathbf{H}_{in}$ and $\mathbf{H}_{out}$ from being singular. However, after the FE assembly has been solved for nodal displacements, those missing terms can easily be recovered. The intraelement field $\dot{\mathbf{U}}$ in an element may be argumented by the rigid body modes:

$$\mathbf{u} = \mathring{\mathbf{u}} + \mathbf{N}_{in}\mathbf{c}_{in} + \bar{\mathbf{N}}_{in}\bar{\mathbf{c}}_{in} \tag{101}$$

$$\mathbf{w} = \mathring{\mathbf{w}} + \mathbf{N}_{out}\mathbf{c}_{out} + \bar{\mathbf{N}}_{out}\bar{\mathbf{c}}_{out} \tag{102}$$

where

$$\bar{\mathbf{N}}_{in} = \begin{bmatrix} 1 & 0 & y \\ 0 & 1 & -x \end{bmatrix}, \quad \bar{\mathbf{N}}_{out} = \begin{bmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{103}$$

Using a least square procedure to match with the nodal parameters $\mathbf{U}_i$ at corner nodes:

$$\sum_i [(\dot{U}_{1i} - \tilde{U}_{1i})^2 + (\dot{U}_{2i} - \tilde{U}_{2i})^2] = \min \tag{104}$$

$$\sum_i [(\dot{W}_i - \tilde{W}_i)^2/a^2 + (\dot{\varphi}_{xi} - \tilde{\varphi}_{xi})^2 + (\dot{\varphi}_{yi} - \tilde{\varphi}_{yi})^2] = \min \tag{105}$$

where Σ extends over all nodes of the element, which yields

$$\mathbf{c}_{in} = \mathbf{R}_{in}^{-1}\mathbf{r}_{in}, \quad \mathbf{c}_{out} = \mathbf{R}_{out}^{-1}\mathbf{r}_{out} \tag{106}$$

where

$$\mathbf{R}_{in} = \sum_i \begin{bmatrix} 1 & 0 & y_i \\ 0 & 1 & -x_i \\ y_i & -x_i & x_i^2 + y_i^2 \end{bmatrix} \tag{107}$$

$$\mathbf{r}_{in} = \sum_i \begin{Bmatrix} \tilde{U}_{1i} - \dot{U}_{1i} \\ \tilde{U}_{2i} - \dot{U}_{2i} \\ (\tilde{U}_{1i} - \dot{U}_{1i})y_i - (\tilde{U}_{2i} - \dot{U}_{2i})x_i \end{Bmatrix} \tag{108}$$

$$\mathbf{R}_{out} = \frac{1}{a^2}\sum_i \begin{bmatrix} a^2 & x_i a^2 & y_i a^2 \\ x_i & x_i^2 + a^2 & x_i y_i \\ y_i & x_i y_i & y_i^2 + a^2 \end{bmatrix} \tag{109}$$

$$\mathbf{r}_{out} = \sum_i \begin{Bmatrix} \tilde{W}_i - \dot{W}_i \\ x_i(\tilde{W}_i - \dot{W}_i)/a^2 + \tilde{\varphi}_{xi} - \dot{\varphi}_{xi} \\ y_i(\tilde{W}_i - \dot{W}_i)/a^2 + \tilde{\varphi}_{yi} - \dot{\varphi}_{yi} \end{Bmatrix} \tag{110}$$

The above outlines general formulation followed and is applicable to the element model considered here. But the use of this formulation should obey the following rule. This rule is described that the lower limit on the number m_{in} and m_{out} of internal parameters $\mathbf{c}_{in}$ and $\mathbf{c}_{out}$ is prescribed by the necessary (but not sufficient) rank condition (Jirousek and Venkatesh, 1992)

$$m_{in} \geqslant (NDOF)_{in} - 3, \quad m_{out} \geqslant (NDOF)_{out} - 3$$

where $(NDOF)_{in}$ and $(NDOF)_{out}$ are the size of element DOF for in-plane and out-of-plane displacements, respectively.

2.7. *Iterative scheme*

Since $\mathbf{P}_{in}$ and $\mathbf{P}_{out}$ contain the unknown variables $(\dot{U}_1 \dot{U}_2 \dot{W})$, an iterative procedure is, thus, required. Before describing the scheme, however, let us study some properties of $\mathbf{P}_{in}$. It can be seen from the definition of $\mathbf{P}_{in}$ that it only depends upon $\dot{\mathbf{W}}$. So only an initial value $\mathbf{W}^0$ is required. As long as the value of $\dot{\mathbf{W}}$ in Ω is known, we can calculate the pseudo-load $\mathbf{P}_{in}$, and then all of unknown variables in (95) are in-plane displacements $(\dot{U}_1 \dot{U}_2)$. We may solve (95) for them. As a consequence, $\mathbf{P}_{out}$ can be calculated from the current values of $(\dot{U}_1 \dot{U}_2 \dot{W})$. An iterative scheme may be proposed according to the above analysis. Specifically, suppose that U_1^k, U_2^k and W^k stand for kth approximations, which can be obtained from the preceding cycle of iteration. The $(k+1)$ solution may be evaluated as follows

(a) Assume the initial value $\dot{W}^0$ and W^0 in Ω if the current loading step is not the first one, but $(k+1)$th step, $\dot{W}^0$ and W^0 in Ω may be taken as $\dot{\mathbf{W}}^k$ and $\mathbf{W}^k$, where $\dot{\mathbf{W}}^k$ and $\mathbf{W}^k$ stand for the incremental and the total deflection at kth loading step, respectively.
(b) Enter the iterative cycle for $i = 1, 2, \ldots$. Calculate $\mathbf{P}_{in}$ in (95) by means of (99), solve (95) for the nodal displacement vector $\mathbf{d}_{in}^{(i)}$, and then determine the values of $\dot{U}_1^{(i)}$ and $\dot{U}_2^{(i)}$ in Ω.
(c) Calculate $\mathbf{P}_{out}$ using the current values of U, then solve (96) for $\mathbf{d}_{out}^{(i)}$ and determine the value of $\dot{W}^{(i)}$ in Ω.
(d) If $\varepsilon_i = [(\mathbf{d}^{(i)})^{\mathrm{T}}\mathbf{d}^{(i)} - (\mathbf{d}^{(i-1)})^{\mathrm{T}}\mathbf{d}^{(i-1)}]/(\mathbf{d}^{(i-1)})^{\mathrm{T}}\mathbf{d}^{(i-1)} \leqslant \varepsilon$ (ε is a convergence tolerance), proceed to the next loading step and calculate

$$\mathbf{U}^{k+1} = \mathbf{U}^{(k)} + \dot{\mathbf{U}}^{(i)}, \quad \dot{\mathbf{U}}^{k+1} = \dot{\mathbf{U}}^{(i)}$$

otherwise, set

$$\dot{W}^0 = \dot{W}^{(i)}, \quad W^0 = W^k + \dot{W}^{(i)}$$

and go back to step (b).

3. NUMERICAL APPLICATIONS

Since the main purpose of this paper is to outline the basic principles of the proposed method, the assessment will be limited to three simple examples. In order to allow for comparisons with other solutions appearing in references (Katsikadelis, 1991; Ng and Das, 1986; Smaill, 1991), the obtained numerical results are limited to a circular plate on a Winkler-type foundation, a skew sandwich plate on a Winkler-type foundation and an annular plate on a Pasternak-type foundation. To study the convergence properties of the proposed method, three meshes for the solution domain are used in each example. In all the calculations, the convergence tolerance is $\varepsilon = 0.0001$.

Example 1 : *a circular plate on a Winkler-type foundation.* Consider a uniformly loaded circular plate resting on a Winkler-type foundation, and with radius a and clamped movable edges (i.e., $\dot{W} = \dot{\varphi}_n = \dot{\varphi}_s = \dot{N}_n = \dot{N}_{ns} = 0$). Some parameters for the problem are assumed as

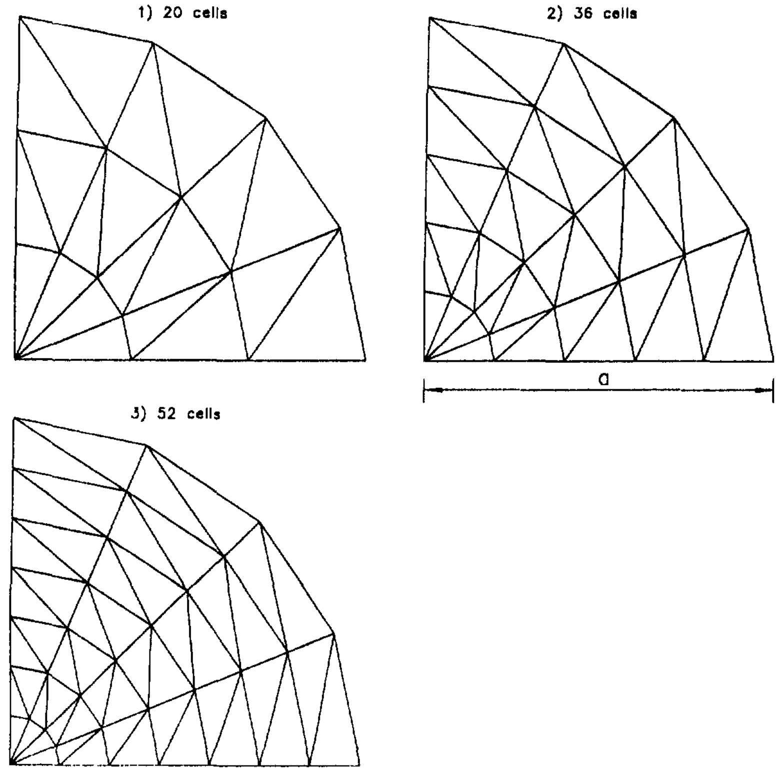

Fig. 3. Three element meshes for example 1.

$$a/t = 50 \quad v = 0.3 \quad k_w a^4/D = 100 \quad Q = qa^4/Et^4 = 15$$

A quadrant of the plate is modeled by three meshes (see Fig. 3) and the loading step is taken as $\Delta Q = 1$. Table 2 shows the deflection $\bar{w}(\bar{w} = W/t)$ along the radius of the plate and compares with the boundary element solution (Katsikadelis, 1991). Table 3 shows the results of deflection $\bar{w}$ vary with M, here M is the number of hierarchic degrees of freedom. In all three examples, the hierarchic degrees of freedom were defined by:

$$M = 1 \Rightarrow p_1, \quad M = 3 \Rightarrow p_1, q_1, r_1,$$
$$M = 5 \Rightarrow p_1, q_1, r_1, a_1, b_1,$$
$$M = 6 \Rightarrow p_1, q_1, r_1, a_1, b_1, p_2$$

Table 2. Deflection $\bar{w}$ along the radius r in the circular plate ($M = 0$)

	r/a	0.098	0.304	0.562	0.800	0.960
	20 cells	1.090	0.948	0.581	0.169	0.009
HT FE	36	1.106	0.957	0.588	0.174	0.008
	52	1.110	0.962	0.591	0.180	0.008
Katsikadelis		1.108	0.961	0.592	0.179	0.009

Table 3. Deflection $\bar{w}$ vs M for example 1 (36 cells)

M	0	1	3	5	6	8	10
$r/a = 0.098$	1.106	1.108	1.110	1.110	1.112	1.112	1.112
0.304	0.957	0.960	0.961	0.961	0.964	0.964	0.965
0.562	0.588	0.591	0.592	0.592	0.594	0.594	0.593
0.800	0.174	0.177	0.177	0.179	0.180	0.181	0.181
0.960	0.008	0.009	0.008	0.008	0.007	0.007	0.007

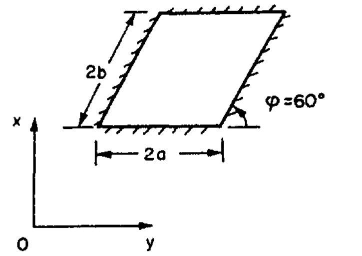

Fig. 4. A Cl 60° skew sandwich plate resting on an elastic foundation.

$$M = 8 \Rightarrow p_1, q_1, r_1, a_1, b_1, p_2, q_2, r_2$$

$$M = 10 \Rightarrow p_1, q_1, r_1, a_1, b_1, p_2, q_2, r_2, a_2, b_2$$

Example 2: *a* 60° *skew sandwich plate on a Winkler-type foundation.* The skew plate is clamped immovable on all edges (i.e., $\dot{U}_n = \dot{U}_s = \dot{W} = \dot{\varphi}_n = \dot{\varphi}_s = 0$ on the whole boundary) shown in Fig. 4. Some initial data are

$$v = 0.32,\quad t = 0.635\,\text{mm},\quad h = 25.4\,\text{mm},\quad a = b = 508\,\text{mm},\quad z = t+h,$$

$$G_c = 6.89\,\text{MPa},\quad Q = 12a^3(1-v^2)q/(tz^2E),\quad K = 12a^3k_w(1-v^2)/(ztE)$$

where $2a$ stands for the length of the skew plate (see Fig. 4). The plate is equally divided into $N \times N$ ($N = 2, 4$ and 6) elements. The loading step is $\Delta Q = 12.5$. Table 4 compares the present results with those obtained by Ng and Das (1986) in which the values were obtained from Fig. 11 in their paper. Table 5 lists the relationship between central deflection W_c/h and M.

Example 3: *an annular plate on a Pasternak-type foundation.* The annular plate is subjected to a uniform distributed load $q(Q = ga^4/E^4)$ and resting on a Pasternak-type foundation. The inner boundary of the plate was in a free edge condition while the out boundary condition was clamped immovable. Some initial data used in the example are given by

Table 4. Deflection W_c/h vs Q for the sandwich plate ($M = 0$)

	Q	25	50	75	100	125
	2×2	0.591	0.862	1.026	1.161	1.283
HT FE	4×4	0.598	0.871	1.037	1.178	1.298
	6×6	0.603	0.875	1.045	1.185	1.304
Ng and Das		0.60	0.87	1.05	1.18	1.30

Table 5. Central deflection W_c/h vs M for example 2 (4×4)

M	0	1	3	5	6	8	10
$Q = 25$	0.598	0.601	0.603	0.603	0.605	0.606	0.606
50	0.871	0.874	0.875	0.875	0.876	0.976	0.877
75	1.037	1.040	1.042	1.043	1.047	1.049	1.050
100	1.178	1.183	1.183	1.184	1.187	1.188	1.188
125	1.298	1.301	1.302	1.302	1.306	1.307	1.308

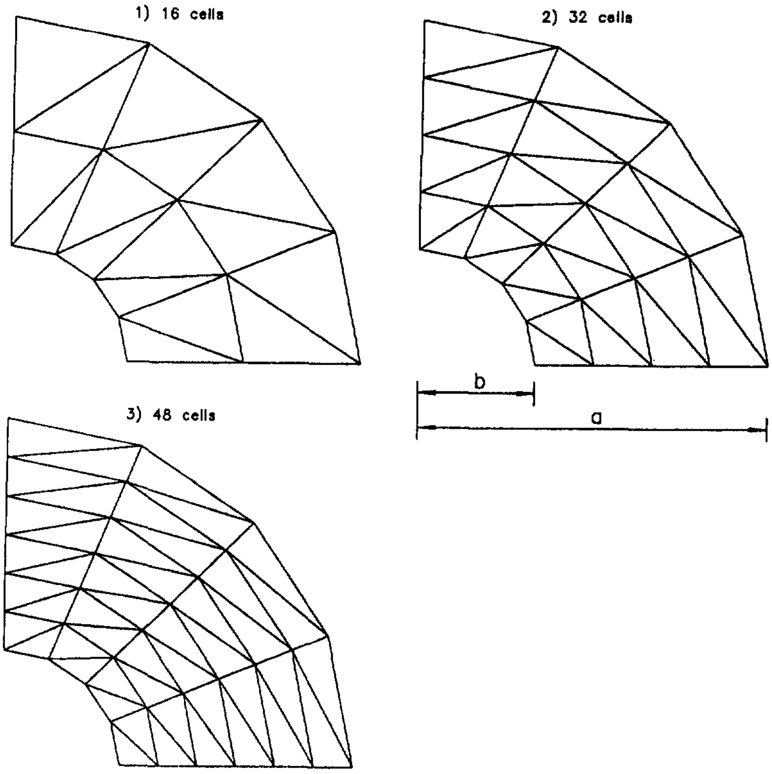

Fig. 5. Three element meshes in example 3.

$$g = G_p a^2/E = 1, \quad K = k_p a^4/E = 5, \quad b/a = 1/3, \quad \nu = 1/3$$

where a is the outer radius, and b the inner radius (see Fig. 5). In the example, a quarter of the plate is modeled by three meshes shown in Fig. 5. The loading step is taken as $\Delta Q = 5$. Some results obtained by the proposed method are listed in Tables 6 and 7.

It can be seen from the above tables that the results obtained by the present method agree well with the existing results. As expected for all examples, it is also found from Tables 2, 4 and 6 that the proposed formulation yields converging values along with refinement of the element meshes. The results in Tables 3, 5 and 7 show that the hierarchic DOF p_i q_i r_i are more important than in-plane DOF a_i b_i in these examples. In the course of computations, convergence was achieved with about 8 iterations for example 1, 15 iterations for example 2 and 12 iterations for example 3 at each load step.

Table 6. Maximum deflection W_m/t vs Q for example 3 ($M = 0$)

	Q	10	15	20	25	30
	16 cells	0.491	0.725	0.920	1.082	1.227
HT FE	32	0.508	0.732	0.929	1.095	1.238
	48	0.513	0.738	0.935	1.105	1.243
Smaill		0.51	0.74	0.93	1.10	1.24

Table 7. Maximum deflection W_m/t vs M for example 3 (32 cells)

M	0	1	3	5	6	8	10
$Q = 10$	0.508	0.512	0.513	0.515	0.518	0.518	0.519
15	0.732	0.735	0.736	0.736	0.739	0.740	0.742
20	0.929	0.933	0.935	0.936	0.940	0.942	0.943
25	1.095	1.099	1.099	1.100	1.103	1.103	1.104
30	1.238	1.242	1.244	1.244	1.248	1.250	1.251

4. CONCLUSIONS

A HT FE model with p-method capabilities has been presented for nonlinear analysis of Reissner plates on an elastic foundation. As far as we know, previous HT FE results only deal with the linear problems. However, an elastic plate can undergo deformations that are obvious to the unaided eye. Often such deformations are nonlinear in the sense that the displacements at a given point on the plate are not proportional to the magnitude of the applied load, and therefore a nonlinear analysis is required to treat such a category of behavior. The basic contributions of the paper are: (i) a T-complete set of homogeneous solutions for Reissner-Mindlin plates on elastic foundations has been derived and used to represent the intraelement displacement and rotation fields; (ii) some modifications on nonlinear boundary equations have been proposed to make the related derivation tractable. The practical efficiency of these modifications has been assessed through three numerical examples which have shown that the new p-element is robust, more accurate in terms of the number of unknowns and of the computational effort; (iii) a modified variational principle for nonlinear analysis of thick plates on an elastic foundation has been presented and used to derive HT FE formulation. The attractiveness of this new element can further be enhanced by the implementation of special, globally based load terms which enable various discontinuous loads (line loads, concentrated loads, etc.) to be accurately handled without tedious and expensive mesh adjustment.

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